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Newtonian Universes Expanding or Contracting with Shear and Rotation

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Abstract

Newtonian cosmology is examined fundamentally and an approach initiated by J. V. Narlikar (1963) is pursued in considerable detail. Homogeneous models expanding with spin and shear are shown unequivocally to possess a singularity. Nevertheless in the presence of sufficient spin they can pass through a well-behaved density maximum at some other epoch. Discrepancies between the conclusions of Narlikar and those of Heckmann & Shucking (1955, 1956) are cleared up. Representative solutions, including models tentatively matched to the actual universe are illustrated.

1. *Basic Postulates and Equations*

(a) *Introduction*

In the absence of explicit solutions of Einstein's equations for expanding rotating world models the Newtonian approach is likely to give useful insight. The problem of a universe expanding with shear and rotation was first tackled from the Newtonian point of view by Heckmann & Shucking (1955, 1956), following the early work of McCrea & Milne (1934) for isotropic Newtonian models. Heckmann and Shucking concluded that the occurrence of a singularity could be avoided in Newtonian models provided that a universal spin was present. This was later contested by Narlikar (1963) who claimed that it was not possible to get non-singular oscillating models on the basis of Newtonian theory.

The present work confirms that Narlikar was right on the basis of his own assumptions which appear to be physically more realistic than those of Heckmann and Shucking. However Narlikar's case was far from being made and indeed incorrect at a vital point. Accepting Narlikar's basic approach we pursue the analysis of Newtonian universes to considerably greater depths and examine the evolutionary properties of representative solutions. The nature of the Newtonian singularity is discussed in some detail, and the discrepancy between Narlikar's qualitative conclusion and that of Heckmann and Shucking is explained. Many of the results are of current interest in cosmology and specific models are tentatively matched to the present properties of the actual universe.†

(b) *Fundamentals of the Present Approach*

Our Newtonian universe will be taken to be infinite, homogeneous and euclidean, containing pressureless dust (fundamental particles) of uniform density $\rho(t)$ at Newtonian time t. We take any such fundamental particle O to be the origin of a cartesian frame of reference, of coordinates x_i ($i = 1, 2, 3$). A similar system set up at the fundamental particle O', with axes parallel to those at O, is related by the euclidean transformation

$$x_i = X_i + x_i' \tag{1.1}$$

where X_i are the coordinates of O' in O's system. Also, the velocity components of a test particle P relative to the two fundamental frames will satisfy the relation of Newtonian kinematics:

$$\dot{x}_i = \dot{X}_i + \dot{x}_i' \tag{1.2}$$

a dot indicating differentiation with respect to t.

We now suppose that the postulated homogeneity includes a 'cosmological principle' on the following basis. At any epoch t the motion of world matter relative to an observer comoving with O is to be the same as that relative to an observer at O'. Hence, if $\dot{x}_i = v_i(x_j)$, then $\dot{x}_i' = v_i(x_j')$. From (1.1), (1.2) it then follows that

$$v_i(X_j + x_j') = v_i(X_j) + v_i(x_j') \tag{1.3}$$

so that $v_i(x_j)$ must be a linear function of x_j, the coefficients being in general functions of t. That is

$$\dot{x}_i = H_{ij}(t)\, x_j \tag{1.4}$$

In general the motion given by (1.4) will be subject to shear, the shear tensor $q_{ij}(t)$ being

$$q_{ij} = \tfrac{1}{2}(H_{ij} + H_{ji}) - \tfrac{1}{3}H_{kk}\,\delta_{ij} = q_{ji} \tag{1.5}$$

with

$$q_{ii} = 0. \tag{1.6}$$

There will also in general be spin, the spin tensor $\omega_{ij}(t)$ being

$$\omega_{ij} = \tfrac{1}{2}(H_{ij} - H_{ji}) = -\omega_{ji} \tag{1.7}$$

It follows from (1.4) that the acceleration $\ddot{x}_i$ relative to the typical fundamental frame of origin O is also linear in x_j, having the form

$$\ddot{x}_i = A_{ij}\, x_j \tag{1.8}$$

where

$$A_{ij} = \dot{H}_{ij} + H_{ik} H_{kj} \tag{1.9}$$

† A brief report of this work has already appeared (Davidson and Evans, 1971).

In applying Newtonian mechanics and gravitation to the infinite system postulated, one approach can now be made as follows. If we assume that a unique Newtonian frame, or absolute space, exists then in accordance with (1.8) we can find a fundamental particle O' which at time t has zero acceleration relative to such a frame. Thus if absolute space has acceleration f_i in O's system, we can find an O' of coordinates X_i such that

$$A_{ij} X_j = f_i$$

provided the motion of the dust system is not degenerate, i.e., $\det A_{ij} \neq 0$. Because the dust system is infinite and homogeneous it is at least plausible to assume that O' will always have zero acceleration in absolute space. We can then re-label O' as O, a fundamental particle lying permanently at the origin of the unique Newtonian frame in which the accelerations and forces are absolute.

Alternatively, one might assume that any fundamental frame is a valid Newtonian frame, again because of the postulated homogeneity of the dust system and its infinite extent. This would mean that Newtonian frames could exist in relative acceleration. In turn this would require the surrender of the concepts of absolute acceleration and absolute force, as follows. In any pair of fundamental frames treated as Newtonian, the second law of motion would hold in the form

$$F_i = \ddot{x}_i, \quad F_i' = \ddot{x}_i' \tag{1.10}$$

where F_i and F_i' are the Newtonian forces per unit mass. Moreover, the accelerations of a test particle in two such frames would be connected by

$$\ddot{x}_i = \ddot{X}_i + \ddot{x}_i' \tag{1.11}$$

on differentiating (1.2). Therefore, assuming the usual invariant and constant character of Newtonian mass the forces F_i, F_i' in the two frames would stand in the relation

$$F_i = \ddot{X}_i + F_i' \tag{1.12}$$

In this alternative approach to Newtonian cosmology, therefore, the fictitious forces $\ddot{X}_i$ would play an important role in preserving the equivalence of the dynamical laws in all Newtonian frames. In this paper the fundamental particles themselves will be assumed to be under mutual gravitational forces only. Hence in the present observable universe, even at the distance of the observable horizon associated with the Hubble law, the fictitious force $\ddot{X}_i$ would have a magnitude of about 10^{-9} cgs only. The non-uniqueness of the cosmic Newtonian frame would therefore have negligible effect on Newtonian laboratory physics at the present epoch. On the cosmic scale of the dust medium itself the fictitious force $\ddot{X}_i$ is of course not negligible, since it is of the same order and indeed of the same nature as the gravitational force F_i that we shall be considering.

(c) *Equations of Motion*

Let F_i be the Newtonian gravitational field intensity at the point x_i, either in the unique Newtonian frame first considered in (b) or alternatively in any fundamental Newtonian frame. Then Newton's second law of motion in that frame is the Euler equation for the fluid of dust particles, having the form

$$F_i = \ddot{x}_i = A_{ij} x_j \tag{1.13}$$

in accordance with (1.8).

Solutions for the fluid motion will be derived in the present paper for the special case when A_{ij} is an isotropic tensor. Since the universe is assumed infinite and homogeneous, the combination of the inverse square law of direct attraction and the mass symmetry about the Newtonian origin make it plausible that the gravitational field should also be symmetric about the origin. In this case, by the Gauss theorem

$$F_i = -\tfrac{4}{3}\pi G\rho x_i \tag{1.14}$$

G being the Newtonian gravitational constant. Hence

$$A_{ij} \equiv \dot{H}_{ij} + H_{ik} H_{kj} = -\tfrac{4}{3}\pi G\rho\, \delta_{ij} \tag{1.15}$$

If we add the continuity equation for a uniform fluid we get

$$\frac{\dot{\rho}}{\rho} + \dot{x}_{i,i} = 0$$

where $,i$ indicates partial differentiation with respect to x_i. That is

$$\frac{\dot{\rho}}{\rho} + H_{ii} = 0 \tag{1.16}$$

Elimination of ρ between (1.15) and (1.16) provides nine equations for the nine H_{ij}, and with given initial conditions the problem is then determinate.

As we have stated, the solutions to the Newtonian problem to be given in the present paper are special, to the extent of being based on an isotropic A_{ij} tensor, or a gravitational field that is spherical in the Newtonian reference frame. Nevertheless, the field in any real homogeneous universe seems likely to be nearly spherical in a wide range of cosmic motions which include spin and shear. Essentially this is because the gravitational effect of motion (according to relativity theory) is so much less than that of mass. Therefore our results should provide qualitative insight into the general problem. Zeldovich (1965) and more recently Shikin (1971) have also given Newtonian solutions to the cosmological problem, but neither considered the possibility of spin which we do here.

The work of Heckmann & Shucking (1955, 1956) should be mentioned in the present discussion. It will be noted that in our own analysis there has been no necessity to introduce a Newtonian potential. Heckmann and Shucking used a potential function ϕ satisfying the Poisson equation

$\phi_{,ii} = 4\pi G\rho_{,}$. They also assumed the integrability condition $\phi_{,ij} = \phi_{,ji}$ (curl$F = 0$), but did not have the equivalent of our (1.15). In the absence of boundary conditions in a homogeneous infinite system five other equations were therefore required to fully determine the nine $\phi_{,ij}$. To solve the problem Heckmann and Shucking assumed that the five independent components of the shear tensor q_{ij} could be assigned arbitrarily, and in practice their solutions were obtained on the assumption that $q_{ij} = 0$ for all t. Heckmann and Shucking's approach is therefore quite different from ours. In our present formulation the evolution of shear is determined from the initial conditions and, as we shall show, shear always develops in the presence of spin. In both these respects our results are in agreement with general relativity. Moreover we shall show that under our present formulation the assumption that $q_{ij} = 0$ for all t would violate angular momentum conservation (see Section 2(a)). It is precisely this assumption that is responsible for Heckmann and Shucking's conclusion that spin can prevent a singularity (see Section 1(e)).

(d) *Use of Lagrangian Coordinates*

With given initial conditions the equations (1.4) determine a comoving coordinate system x_i^0 given by

$$x_i = a_{ij}(t)\, x_j^0 \tag{1.17}$$

such that

$$\dot{a}_{ij} = H_{ik}\, a_{kj} \tag{1.18}$$

If t_0 is a given value of t we can in fact arrange that

$$a_{ij}^0 \equiv a_{ij}(t_0) = \delta_{ij} \tag{1.19}$$

so that the x_i^0 are the values of x_i for the given particle at $t = t_0$ (Lagrangian coordinates). Equations (1.18), (1.19) then uniquely determine the a_{ij} in terms of the H_{ij}, yielding the Eulerian coordinates x_i of the particle.

Denoting the determinant of the a_{ij} by Δ and differentiating with respect to t we find that

$$\dot{\Delta}/\Delta = H_{ii} \tag{1.20}$$

Combining (1.20) with (1.16) we get on integration

$$\rho\Delta = \text{constant} = \rho_0\, \Delta_0 = \rho_0 \tag{1.21}$$

Here $\rho_0 \equiv \rho(t_0)$ and we have set $\Delta_0 \equiv \Delta(t_0) = 1$ in accordance with (1.19). Finally, differentiating (1.18) and using (1.15) we derive the differential equation for the a_{ij}

$$\ddot{a}_{ij} = -\tfrac{4}{3}\pi G\rho_0 \frac{a_{ij}}{\Delta} \tag{1.22}$$

This equation was first given by Narlikar (1963).

(e) *Relations Involving the Spin and Shear*

If we differentiate (1.20) and use the contracted form of (1.15) we obtain the relation

$$\frac{\ddot{\Delta}}{\Delta}-\frac{\dot{\Delta}^2}{\Delta^2}+4\pi G\rho+H_{ij}H_{ji}=0 \tag{1.23}$$

From (1.5) and (1.7) we have also

$$H_{ij}=q_{ij}+\omega_{ij}+\frac{1}{3}\frac{\dot{\Delta}}{\Delta}\delta_{ij} \tag{1.24}$$

Hence

$$H_{ij}H_{ji}=q_{ij}q_{ij}-\omega_{ij}\omega_{ij}+\frac{1}{3}\frac{\dot{\Delta}^2}{\Delta^2}$$

or

$$H_{ij}H_{ji}=q^2-2\omega^2+\frac{1}{3}\frac{\dot{\Delta}^2}{\Delta^2} \tag{1.25}$$

Here we define the spin and shear invariants ω and q, both $\geqslant 0$, by

$$\omega_{ij}\omega_{ij}=2\omega^2 \tag{1.26}$$

$$q_{ij}q_{ij}=q^2 \tag{1.27}$$

respectively. The first definition is of course consistent with the interpretation of ω as the magnitude of the spin vector ω_i where

$$\omega_i=\tfrac{1}{2}e_{ijk}\omega_{jk} \tag{1.28}$$

e_{ijk} being the Levi–Civita tensor.

If we define also a variable $R(t)$ by

$$R=\Delta^{1/3}=\left(\frac{\rho_0}{\rho}\right)^{1/3} \tag{1.29}$$

then $R(t)$ gives a measure of the variation of the linear dimensions of a comoving volume element, or the inverse cube of the mass density. Substitution of (1.25) and (1.29) into (1.23) now yields the relation

$$\frac{3\ddot{R}}{R}=2\omega^2-q^2-\frac{4\pi G\rho_0}{R^3} \tag{1.30}$$

Equation (1.30), which was also derived by Heckmann & Shucking (1955, 1956) from their own premises, is the Newtonian analogue of the relativity equation obtained for dust by Raychaudhuri (1955). It shows that whereas shear assists gravity in a possible convergence towards a singularity ($R\to 0$) spin has the opposite effect. The outcome clearly cannot be decided by this equation alone. A further relation was obtained by Heckmann and Shucking from the integrability conditions for their potential function ϕ viz. $\phi_{,ij}=\phi_{,ji}$. The same result follows however from our more specific condition (1.15) (which as already stated was not part of the theory of Heckmann and Shucking). Combining (1.15) with the equation got by interchanging i and j, and making use of (1.24) and (1.28) we get after some calculation

$$\frac{d}{dt}(R^2\omega_i) = R^2 q_{ij}\omega_j \tag{1.31}$$

An important point now arises. In their specific models Heckmann and Shucking assumed that $q_{ij} = 0$, so that $w_i = c_i/R^2$, where c_i is a constant vector. Substitution into (1.30) then shows that $\ddot{R}$ will be positive when R is sufficiently small, so that apparently a singularity ($R = 0$) is prevented because of spin. In our own approach the arbitrary assumption $q_{ij} = 0$ cannot be made; the evolution of q_{ij} is determined once the initial conditions are set. In our theory ω_i therefore varies in a different way. As a result total collapse in at least one direction cannot be prevented (see Section 2(c)).

From (1.30) we also note the following. Having already set $\Delta_0 = 1$ then (1.29) requires that $R_0 = 1$. Suppose now that the arbitrary epoch t_0 happens to be a stationary point of Δ, and therefore of R and of the density ρ, so that $\dot{\Delta}_0 = 0 = \dot{R}_0$. Then (1.30) shows that at such a point

$$\dot{\Delta}_0 = 0, \qquad \ddot{\Delta}_0 = 3\ddot{R}_0 = 2\omega_0{}^2 - q_0{}^2 - 4\pi G\rho_0 \tag{1.32}$$

It follows that if the spin vanishes at the stationary point then the point can only be a maximum of Δ, or of R. Also, if there is a minimum of Δ or of R at some epoch then the spin cannot vanish there. This indication that the mass density can go through a well-behaved maximum in the presence of sufficient spin is clearly of great dynamical importance. We shall see however that this possibility does not prevent a singularity taking place at some other epoch (see Section 2(f)).

We can also obtain an expression for $\dddot{\Delta}_0 = 3\dddot{R}_0$ at the stationary point by first differentiating (1.23) and using (1.15), to find

$$\dddot{\Delta}_0 = 2(H_{ij}H_{jk}H_{ki})_0$$

at the stationary point $t = t_0$. To evaluate the invariant on the right we express H_{ij} in terms of spin and shear by means of (1.24) and then refer the result to the principal axes of shear at the instant $t = t_0$. If Q_1, Q_2, Q_3 are the principal components of shear, and Ω_1, Ω_2, Ω_3 the components of spin along the principal axes, then reduction gives

$$\dot{\Delta}_0 = 0,\ \dddot{\Delta}_0 = 3\dddot{R}_0 = 6(Q_1 Q_2 Q_3 + Q_1\Omega_1{}^2 + Q_2\Omega_2{}^2 + Q_3\Omega_3{}^2)_0. \tag{1.33}$$

This result will be used in (f).

(f) *A Fourth Order Differential Equation for Δ*

Following Narlikar (1963) we introduce a dimensionless variable τ related to the Newtonian time t by

$$\tau = (\tfrac{4}{3}\pi G\rho_0)^{1/2}\, t \tag{1.34}$$

Denoting differentiation with respect to τ by a prime, equation (1.22) may be written:

$$a''_{ij} = -\frac{a_{ij}}{\Delta} \tag{1.35}$$

Successive differentiation of the determinant $\Delta = \det a_{ij}$, using (1.35) whenever second order derivatives appear, leads to Narlikar's equation for Δ:

$$\Delta^2 \Delta'''' + 7\Delta\Delta'' - 4\Delta'^2 + 9\Delta = 0 \tag{1.36}$$

We note that (1.36) is unchanged on putting $-\tau$ for τ so that the motion is time-reversible, as expected for a dust model in Newtonian theory. Also, since the differential equation is fourth order a solution is completely determined if at the initial time ($t = t_0$, $\tau = \tau_0$, say) Δ_0, Δ_0', Δ_0'', Δ_0''' are specified. In particular, if $\tau = \tau_0$ is a stationary point and it happens that the shear vanishes there we can conclude from (1.33) that all odd-ordered derivatives will vanish at $\tau = \tau_0$, so that in this case the motion is symmetrical about the epoch $\tau = \tau_0$. In the general case, when $\Delta_0''' \neq 0$, the motion will not be symmetrical about a peak or trough in the $\Delta - \tau$ curve.

Narlikar's investigation of the Newtonian models was specifically directed to finding whether they necessarily contained a singularity. His examination was made on the basis of equation (1.36) only. Since a general solution in closed form was not available for this equation Narlikar explored the possible $\Delta - \tau$ development by numerical methods, with varying initial conditions. In doing so he missed the existence of models whose evolution includes procession from a maximum to a minimum of Δ, and vice versa, the assumed absence of which formed a central part of his case for the inevitability of a singular state. In fact study of (1.36) alone obscures the details of the collapse process. In our present work we wish to follow the evolution of the internal structure of Δ, namely the a_{ij}. This is given by (1.35) and it is on this equation that we shall concentrate our main attention. By considering a special type of solution of this equation we shall incidentally be able to show that the general solution to Narlikar's equation (1.36) always contains a singularity, where $\Delta \to 0$, $\rho \to \infty$.

2. *An Axisymmetric Solution*

(a) *Special Variables for a Model Spinning about the x_3 Axis*

The basic equation to be satisfied is

$$a''_{ij} = -\frac{a_{ij}}{\Delta} \tag{2.1}$$

Consider a model for which the x_3 axis is both spin-axis and axis of symmetry of the motion. This can be described by

$$a_{ij} = \begin{bmatrix} x & -u & 0 \\ u & x & 0 \\ 0 & 0 & z \end{bmatrix} \tag{2.2}$$

such that $a_{ij}^0 = \delta_{ij}$ (epoch $\tau = \tau_0$). Equations (2.1) and (1.18) (replacing a dot with a prime) lead to the maintenance of the form (2.2) and an associated H_{ij} given by

$$H_{ij} = \begin{bmatrix} H & -\omega & 0 \\ \omega & H & 0 \\ 0 & 0 & K \end{bmatrix} \tag{2.3}$$

where

$$\left. \begin{aligned} H &= \frac{xx' + uu'}{x^2 + u^2} \\ &= \frac{\eta'}{\eta} \quad \text{where } \eta^2 = x^2 + u^2 \end{aligned} \right\} \tag{2.4}$$

$$K = \frac{z'}{z} \tag{2.5}$$

$$\omega = \frac{xu' - x'u}{x^2 + u^2} \tag{2.6}$$

The Eulerian coordinates of the particle which at $\tau = \tau_0$ was at x_i^0 are given by

$$\left. \begin{aligned} x_1 &= xx_1^0 - ux_2^0 \\ x_2 &= ux_1^0 + xx_2^0 \\ x_3 &= zx_3^0 \end{aligned} \right\} \tag{2.7}$$

$z(\tau)$ is the expansion factor in the x_3 direction. If $r^2 = x_1^2 + x_2^2$ then

$$r = \eta r_0 \tag{2.8}$$

so that $\eta(\tau)$ is the expansion factor in the cylindrical radial direction. The velocity of the particle is given by

$$\begin{aligned} x_1' &= Hx_1 - \omega x_2 \\ x_2' &= \omega x_1 + Hx_2 \\ x_3' &= Kx_3 \end{aligned} \tag{2.9}$$

indicating a universal motion of extension (or contraction) radially and axially together with a twisting about the axis at angular velocity $\omega(\tau)$.

The coordinate axes x_i are evidently the principal axes of shear and the principal components of shear are

$$Q_i(\tau) = \{\tfrac{1}{3}(H-K), \tfrac{1}{3}(H-K), -\tfrac{2}{3}(H-K)\} \tag{2.10}$$

where

$$H - K = \frac{\eta'}{\eta} - \frac{z'}{z} \tag{2.11}$$

To obtain the general value of the spin we may use the fact that angular momentum about the spin axis, following the motion, will be conserved.

This requires

$$\rho\, dx_1\, dx_2\, dx_3\, r^2\, \omega = \rho_0\, dx_1{}^0\, dx_2{}^0\, dx_3{}^0\, r_0{}^2\, \omega_0 \tag{2.12}$$

so that

$$\omega(\tau) = \frac{\omega_0}{\eta^2} \tag{2.13}$$

Referring to (1.31) we then see that we cannot set $q_{ij} = 0$ permanently, as done by Heckmann and Shucking, without inconsistency with (2.13). For under this condition (1.31) would yield $\omega = \omega_0/R^2$ where in the present motion

$$\Delta = R^3 = \eta^2 z \tag{2.14}$$

and the two results for ω would only be reconcilable if $\eta = z$. While the last equality is what would follow from (2.11) as the condition for zero shear we are not in fact permitted to set $\eta = z$ because the equations of motion, discussed in Section 2(b), already determine the evolution of η and z independently.

(b) *Equations of Motion and Integrals*

The equations (2.1) give for the case (2.2) the following equations

$$z'' = -\frac{1}{x^2 + u^2} \tag{2.15}$$

$$x'' = -\frac{x/z}{x^2 + u^2} \tag{2.16}$$

$$u'' = -\frac{u/z}{x^2 + u^2} \tag{2.17}$$

We note that $z'' \leqslant 0$ for all epochs of the motion, which in itself guarantees collapse along the spin axis in the finite past or future. Combining (2.15), (2.16) we get

$$x''z - xz'' = 0$$

and on integrating once there follows

$$x'z - xz' = \text{const} = H_0 - K_0 \tag{2.18}$$

suffix zero indicating as usual evaluations at $\tau = \tau_0$. Similarly we get

$$u'z - uz' = \omega_0 \tag{2.19}$$

$$u'x - ux' = \omega_0 \tag{2.20}$$

For the general value of ω as given by (2.6) we then derive

$$\omega = \frac{u'x - ux'}{x^2 + u^2} = \frac{\omega_0}{\eta^2} \tag{2.21}$$

in agreement with (2.13).

Combining (2.20) with (2.15) we can write

$$\frac{d}{d\tau}\left\{\tan^{-1}\left(\frac{u}{x}\right)\right\} = -\omega_0 z''$$

Hence

$$u = -x\tan\theta,\ \ \theta = \omega_0(z' - K_0) \tag{2.22}$$

Also by (2.19), (2.20) we have

$$\omega_0(z - x) = (K_0 - H_0)\, u \tag{2.23}$$

Whence, assuming $\omega_0 \neq 0$,

$$x = \frac{z}{1 - [(K_0 - H_0)/\omega_0]\tan\theta} \tag{2.24}$$

$$u = -\frac{z\tan\theta}{1 - [(K_0 - H_0)/\omega_0]\tan\theta} \tag{2.25}$$

where

$$\left.\begin{aligned} z'' &= -\frac{(\cos\theta - [(K_0 - H_0)/\omega_0]\sin\theta)^2}{z^2} \\ \theta &= \omega_0(z' - K_0) \end{aligned}\right\} \tag{2.26}$$

The motion is therefore completely determined by the solution of (2.26).

To interpret the important parameter θ, we note that between epoch $\tau = \tau_0$ and the general epoch τ the universe turns right-handedly about the x_3 axis through an angle $\tan^{-1}(u/x)$. Hence θ is the angle through which it turns left-handedly. Since we can assume without loss of generality that $\omega_0 > 0$, it follows from the fact that $z'' < 0$ for all finite x, u that θ is monotonic decreasing as the motion proceeds.

It is now possible to show that *shear always develops in the presence of spin*, even if zero initially. In the model presently considered the shear is zero at $\tau = \tau_0$ if $H_0 - K_0 = 0$. Subsequently, in accordance with (2.11) we have

$$\begin{aligned} H - K &= \frac{(d/d\tau)(\sec\theta)}{\sec\theta}, \qquad \text{using (2.24), (2.25)} \\ &= -\frac{\omega_0 \sin 2\theta}{2z^2}, \ \neq 0 \text{ at general } \tau. \end{aligned}$$

Hence shear develops in the presence of spin, even if zero initially.

From (2.26) we derive an expression for z in terms of θ as follows. We write (2.26) in the form

$$\frac{d\theta}{dz} = -\omega_0{}^2 \frac{(\cos\theta - [(K_0 - H_0)/\omega_0]\sin\theta)^2}{z^2(\theta + \omega_0 K_0)}$$

Hence

$$\frac{1}{z} = 1 + \int_0^\theta \frac{(s + \omega_0 K_0)\, ds}{(\omega_0 \cos s - (K_0 - H_0) \sin s)^2} \tag{2.27}$$

Assuming $K_0 - H_0 \neq 0$, put

$$\tan \alpha = \frac{\omega_0}{|K_0 - H_0|}, \qquad \omega_0 > 0, \qquad 0 < \alpha < \pi/2 \tag{2.28}$$

Then singularities occur in the integral in (2.27) at the values of s given by

$$\left.\begin{array}{lll} s = \alpha, \alpha - \pi & \text{if} & K_0 - H_0 > 0 \\ s = \pi - \alpha, -\alpha & \text{if} & K_0 - H_0 < 0 \\ s = \pi/2, -\pi/2 & \text{if} & K_0 - H_0 = 0 \end{array}\right\} \tag{2.29}$$

The motion evidently depends on two parameters $\omega_0(> 0)$ and $K_0 - H_0$.

(c) *The Case* $K_0 - H_0 > 0$

(i) $K_0 > 0$

First *we integrate backwards* in τ time from the epoch $\tau = \tau_0$. As τ decreases θ increases from zero monotonically towards the value $\theta = \alpha$ when the integrand in (2.27) becomes singular. For the range $0 < \theta < \alpha$ the integral in (2.27) increases monotonically with θ, so that z monotonically decreases. Take two values of θ near $\theta = \alpha$, say θ_0, θ_1 ($\theta_1 > \theta_0$). Put $s = \alpha - \varepsilon$ and let $\alpha - \theta_0 = \varepsilon_0$, $\alpha - \theta_1 = \varepsilon_1$. Then keeping θ_0 fixed and increasing θ_1 towards α we find that

$$\begin{aligned} \int_{\theta_0}^{\theta_1} \frac{(s + \omega_0 K_0)\, ds}{(\omega_0 \cos s - (K_0 - H_0) \sin s)^2} &= -\frac{1}{\omega_0^2} \int_{\epsilon_0}^{\epsilon_1} \frac{(\alpha + \omega_0 K_0 - \varepsilon)\, d\varepsilon}{\{\cos(\alpha - \varepsilon) - \cot \alpha \sin(\alpha - \varepsilon)\}^2} \\ &\sim -\frac{1}{\omega_0^2} (\alpha + \omega_0 K_0) \sin^2 \alpha \int_{\epsilon_0}^{\epsilon_1} \frac{d\varepsilon}{\varepsilon^2} \\ &= \frac{1}{\omega_0^2} (\alpha + \omega_0 K_0) \sin^2 \alpha \left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0} \right) \\ &\to \infty \ \text{ as } \varepsilon_1 \to 0 \end{aligned} \tag{2.30}$$

Hence as $\theta \to \alpha$ the integral in (2.27) $\to \infty$. Therefore z tends monotonically to zero.

Both z' and z'' remain finite as $z \to 0$. In fact since $\theta = \omega_0(z' - K_0) \to \alpha$ we obtain

$$z' \to K_0 + \frac{\alpha}{\omega_0} \tag{2.31}$$

or

$$z \sim \frac{\omega_0{}^2}{(\alpha + \omega_0 K_0)\sin^2 \alpha} (\alpha - \omega_0(z' - K_0)) \tag{2.32}$$

Differentiating we get

$$z' \sim \frac{-\omega_0{}^3 z''}{(\alpha + \omega_0 K_0) \sin^2 \alpha}$$

so that

$$z'' \to \frac{-(\alpha + \omega_0 K_0)^2 \sin^2 \alpha}{\omega_0{}^4} \tag{2.33}$$

It then follows from (2.15) that η also remains finite as $z \to 0$:

$$\eta \to \frac{\omega_0{}^2}{(\alpha + \omega_0 K_0) \sin \alpha} \tag{2.34}$$

Therefore from (2.21) we find that the spin remains finite:

$$\omega \to \frac{(\alpha + \omega_0 K_0)^2 \sin^2 \alpha}{\omega_0{}^3} \tag{2.35}$$

We infer therefore that the nature of the Newtonian singularity in the presence of spin is a collapsed rotating 'pancake', of finite angular velocity and finite expansion.

For the limiting radial shear we find as $\varepsilon \to 0$

$$\frac{\eta'}{\eta} \sim -\frac{(\alpha + \omega_0 K_0) \sin^2 \alpha}{\omega_0{}^3} \log \varepsilon \to +\infty \tag{2.36}$$

and so

$$\eta' \sim -\frac{\sin \alpha}{\omega_0} \log \varepsilon \to +\infty \tag{2.37}$$

For the behaviour of the expansion determinant Δ near the singularity (at $\tau = \tau_*$, say) we now obtain

$$\Delta = \frac{\omega_0{}^3}{(\alpha + \omega_0 K_0) \sin^2 \alpha} (\tau - \tau_*) - 3(\tau - \tau_*)^2 \log (\tau - \tau_*) + \\ + 0(\tau - \tau_*)^2 \tag{2.38}$$

This is to be compared with the well-known behaviour of the isotropic Newtonian model near its point singularity, at which $\Delta \sim (\tau - \tau_*)^2$. The presence of spin therefore has the effect of speeding up the rate of expansion.

Integrating forwards in τ time from $\tau = \tau_0$, we note that θ starting from zero is monotonic decreasing through negative values. Hence the integral in (2.27) is monotonic decreasing (z increases) as long as $\theta > -\omega_0 K_0$. If the value $\theta = -\omega_0 K_0$ is reached then z has reached a maximum ($z' = 0$, $z'' < 0$) and thereafter the integral $\int_0^\theta$ will start to increase and z to decrease. Now

$\theta = -\omega_0 K_0$ will be reached provided z does not become infinite for $\theta > -\omega_0 K_0$. This requires

$$\left.\begin{aligned} &\int_0^{-\omega_0 K_0} \frac{(s+\omega_0 K_0)\,ds}{(\omega_0 \cos s - (K_0 - H_0)\sin s)^2} > -1 \\ \text{and}\quad &\alpha - \pi < -\omega_0 K_0 \end{aligned}\right\} \tag{2.39}$$

In this case θ will fall below the value $-\omega_0 K_0$ until as $\theta \to \alpha - \pi$ a singular state is again approached and $z \to 0$ ($\tau \to \tau_*'$, say). All results at the second singularity are got by putting $\alpha - \pi$ for α.

On the other hand if (2.39) is violated such that

$$\left.\begin{aligned} &\int_0^{-\omega_0 K_0} \leqslant -1 \\ \text{with}\quad &\alpha - \pi < -\omega_0 K_0 \end{aligned}\right\} \tag{2.40}$$

then when $\theta \to \theta_1 \geqslant -\omega_0 K_0$ we shall have $z \to \infty$. Finally if

$$\alpha - \pi \geqslant -\omega_0 K_0 \tag{2.41}$$

then as $\theta \to \theta_1 > \alpha - \pi$ we again have $z \to \infty$.

The asymptotic characteristics of the model when $z \to \infty$ are seen as follows. By (2.24), (2.25) we have at all epochs

$$\eta = \frac{z \sec\theta}{1 - \cot\alpha \tan\theta} \tag{2.42}$$

Accordingly, when $\theta \to \theta_1$, $z \to \infty$ we have

$$\eta \sim \frac{z \sec\theta_1}{1 - \cot\alpha \tan\theta_1}$$

Hence referring to (2.10), (2.11) the components of shear $Q_i \to 0$ when $z \to \infty$. Also, since $\eta \to \infty$ the spin $\omega \to 0$. Thus the model tends to the isotropic state when $z \to \infty$. Moreover, since $z' \to (\theta_1 + \omega_0 K_0)/\omega_0$ it follows that

$$z \sim \frac{(\theta_1 + \omega_0 K_0)}{\omega_0}\tau \qquad \text{as } \tau \to \infty \tag{2.43}$$

while

$$\eta \sim \frac{\sec\theta_1(\theta_1 + \omega_0 K_0)}{\omega_0(1 - \cot\alpha\tan\theta_1)}\tau \tag{2.44}$$

Hence

$$\Delta = \eta^2 z \sim \frac{\sec^2\theta_1(\theta_1 + \omega_0 K_0)^3}{\omega_0{}^3(1 - \cot\alpha\tan\theta_1)^2}\tau^3 \tag{2.45}$$

and the linear expansion factor $R(\tau)$ is asymptotically proportional to τ.

In the case when the model evolves from a singularity to a singularity it necessarily passes through a maximum of the expansion determinant Δ. It will also be possible for a minimum to be present, at $\tau = \tau_0$ say, if at that point the spin relative to shear satisfies the condition.

$$2\omega_0^2 - q_0^2 - 3 > 0 \tag{2.46}$$

in accordance with (1.32), using τ time. Likewise, when the model expands indefinitely from a singular state there may be an intermediate maximum of Δ, followed necessarily by a minimum. Specific models possessing these various characteristics will be illustrated in Section 3.

It will be noted that the total angle through which the universe turns in any of the motions discussed has an upper limit of π radians. The upper limit applies to the twist taking place between two singularities.

(ii) $K_0 < 0$

In this case the motions are essentially similar to those already discussed, the epoch $\tau = \tau_0$ simply being chosen at a different stage in the evolution. However in the case when $z \to \infty$ the model is collapsing from infinite dispersion to a singularity (possibly passing through a minimum of Δ followed by a maximum).

Finally, for the case $K_0 - H_0 > 0$ we integrate (2.27) to express z explicitly in terms of θ, in the form

$$\frac{1}{z} = 1 + \frac{\tan\alpha}{\omega_0^2}\left\{\sin\alpha\cos\alpha\log(\cos\theta - \cot\alpha\sin\theta) + \right.$$
$$\left. + \frac{\theta + \omega_0 K_0}{1 - \cot\alpha\tan\theta} - \theta\sin^2\alpha - \omega_0 K_0\right\} \tag{2.47}$$

to which we add

$$\left.\begin{aligned} x &= \frac{z}{1-\cot\alpha\tan\theta} \\ u &= -\frac{z\tan\theta}{1-\cot\alpha\tan\theta} \\ \eta &= \frac{z\sec\theta}{1-\cot\alpha\tan\theta} \\ \tan\alpha &= \frac{\omega_0}{K_0 - H_0} \\ \theta &= \omega_0(z' - K_0) \end{aligned}\right\} \tag{2.48}$$

(d) *The Case* $K_0 - H_0 < 0$

This case consists of the time reversals of the various models arising under (c). The maximum limits of the motion are now $\theta = \pi - \alpha$ and $-\alpha$, where α is defined by (2.28). As τ increases θ decreases as before.

(e) *The Case* $K_0 - H_0 = 0$

Here the shear at $\tau = \tau_0$ is zero but, as shown in (b), shear subsequently develops in the presence of spin. Referring to (2.24)–(2.26) we have for this case

$$\left.\begin{aligned} x &= z \\ u &= -z\tan\theta \\ z'' &= -\frac{\cos^2\theta}{z^2} \\ \theta &= \omega_0(z' - K_0) \end{aligned}\right\} \tag{2.49}$$

and by (2.27)

$$\frac{1}{z} = 1 + \frac{1}{\omega_0{}^2}\int_0^\theta \frac{(s + \omega_0 K_0)}{\cos^2 s}\,ds \tag{2.50}$$

The singularities of (2.50) occur when $s = \pi/2$, $-\pi/2$ and the possible motions are essentially the same as for the case when $K_0 - H_0 > 0$, with α now set equal to $\pi/2$. Corresponding to the general expression for $1/z$ in (2.47) we now get by integrating (2.50).

$$\frac{1}{z} = 1 + \frac{1}{\omega_0{}^2}\{\log\cos\theta + (\theta + \omega_0 K_0)\tan\theta\} \tag{2.51}$$

If $\tau = \tau_0$ is a stationary value of Δ then $K_0 = H_0 = 0$. In this case both z and x also have stationary values there. In accordance with Section 1(f) the $\Delta - \tau$ graph will then be symmetrical about $\tau = \tau_0$. In this case the motion can only be from a singularity to a singularity since we have $z' = 0$ at $\tau = \tau_0$ and $z'' < 0$ for all τ, the singularities occurring at $\theta = \alpha = \pi/2$ and $\theta = -\alpha = -\pi/2$.

(f) *The Existence of a Singularity in the General Solution of Narlikar's Equation* (1.36)

Narlikar (1963) did not succeed in showing that his equation (1.36) always possessed a singularity. By means of the present axisymmetric solution of the equations (1.35) for the a_{ij} we may show that $\Delta \to 0$, $\rho \to \infty$, at some τ in every solution of (1.36).

Suppose there is a solution of (1.36) without such a singularity. Then a lower bound to Δ must exist. Since Δ is continuous it will assume the lower bound value as a strict minimum. That is the lower bound cannot be approached asymptotically with all derivatives vanishing because equation (1.36) would then be violated. Let the strict minimum occur at $\tau = \tau_0$ and set $\Delta_0 = 1$ as before. Then at the minimum the solution will have $\Delta_0' = 0$, $\Delta_0'' \geqslant 0$. However if $\Delta_0'' = 0$ a minimum requires $\Delta_0''' = 0$ and then (1.36) shows that $\Delta_0'''' = -9$, so that the stationary point would be a maximum and not a minimum. Hence the minimum has

$$\Delta_0 = 1,\ \Delta_0' = 0, \qquad \infty > \Delta_0'' > 0,\ \infty > \Delta_0''' > -\infty \tag{2.52}$$

Because of the argument in Section 1(e) in connection with (1.30) we need not consider cases in which the spin vanishes at $\tau = \tau_0$, which always possess a singularity.

Compare now (2.52) with the axisymmetric solution given by (2.2), (2.3). At a minimum of Δ occurring at $\tau = \tau_0$ we have for the axisymmetric case

$$\left.\begin{aligned} \Delta_0 &= 1 \\ \Delta_0' &= 2H_0 + K_0 = 0 \\ \Delta_0'' &= 2\omega_0^2 - \tfrac{3}{2}K_0^2 - 3 > 0 \\ \Delta_0''' &= 6K_0\left(\omega_0^2 + \frac{K_0^2}{4}\right) \end{aligned}\right\} \qquad (2.53)$$

Clearly, whatever the values of Δ_0'', Δ_0''' specified by (2.52) for the general solution, we can choose by means of (2.53) values of ω_0, K_0 which give the same values of Δ_0'', Δ_0''' in an axisymmetric solution. But we have seen in Section 1(f) that as far as the $\Delta - \tau$ behaviour of any solution of (1.36) is concerned it is completely determined once initial values Δ_0, Δ_0', Δ_0'', Δ_0''' are specified. That is, although the behaviour of the a_{ij} will in general be different from those of the special axisymmetric solution the behaviour of $\Delta = \det a_{ij}$ will be the same. But we have seen that in the axisymmetric solution there is always a point where $\Delta \to 0$, $\rho \to \infty$. Hence the assumption that the solution of (1.36) had no singularity leads to its contradiction; there is always such a singularity.

3. *Representative Examples of the Spinning Axisymmetric Models*

(a) *The case* $H_0 = K_0 = 0$, $\omega_0 = 5$

In this model the reference epoch $\tau = \tau_0$ (taken to be zero) is one of zero shear. It is also a stationary point of Δ since the condition $2H_0 + K_0 = 0$ is satisfied, and the relation (2.46) shows that it is a minimum. In accordance with the analysis of Section 2 the $\Delta - \tau$ graph will therefore be symmetrical about the trough at $\tau = 0$, have two peaks on either side of the trough and plunge to the τ axis at the two singularities which must exist in this case (at $\alpha = \pm\pi/2$). The computed graph is shown in Fig. 1 and clearly demonstrates the well-behaved minimum of Δ, maximum of density ρ, at $\tau = 0$. As predicted by the theory the gradient $\Delta' = d\Delta/d\tau$ at the singularities is non-zero.

The variation of z and η with τ is graphed in Fig. 2. The finite gradient z' at the singularities and the finite radial expansion there relative to the state at $\tau = 0$ is in accordance with the theory. The values of all quantities obtained by numerical integration were found to agree very closely with the theoretical predictions of Section 2. It should be noted that there is a change in sign of the gradient η' just after the first singularity and just before the second. Thus starting from the singularity at negative τ, the

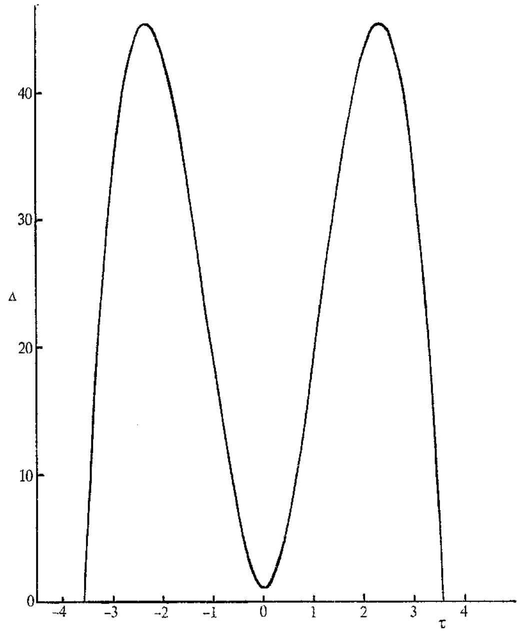

Figure 1.—The graph of the computed relation between Δ and τ for the case $H_0 = K_0 = 0$, $\omega_0 = 5$.

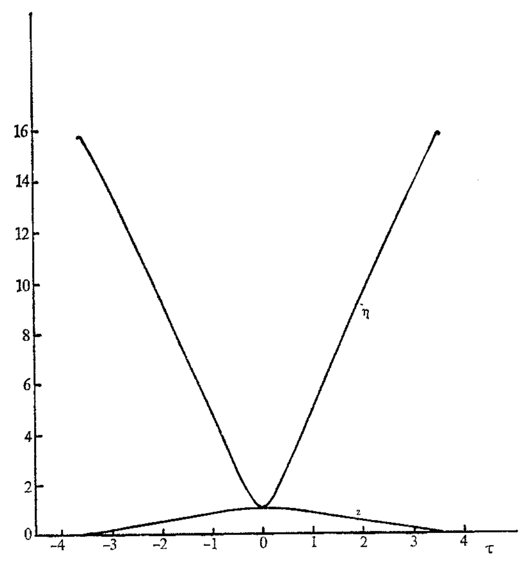

Figure 2.—The computed $\eta - \tau$ and $z - \tau$ relations for $H_0 = K_0 = 0$, $\omega_0 = 5$.

gradient $\eta' = +\infty$ (equation 2.37) but in a very short interval this is reduced to zero and changes smoothly over to being negative. This is a dynamical effect associated with the fact that in the present model the spin at the singularity is relatively small. The intense gravitational field near the singular state therefore very quickly checks the radial expansion and commences to build up the spin in a radial contraction to the maximum of η at $\tau = 0$.

Figure 3 shows for this model the graph of $Y = \Delta'' - \frac{1}{2}(\Delta')^2/\Delta$, against $X = \frac{1}{4}(\Delta')^2/\Delta$, the variables used by Narlikar (1963) in his numerical

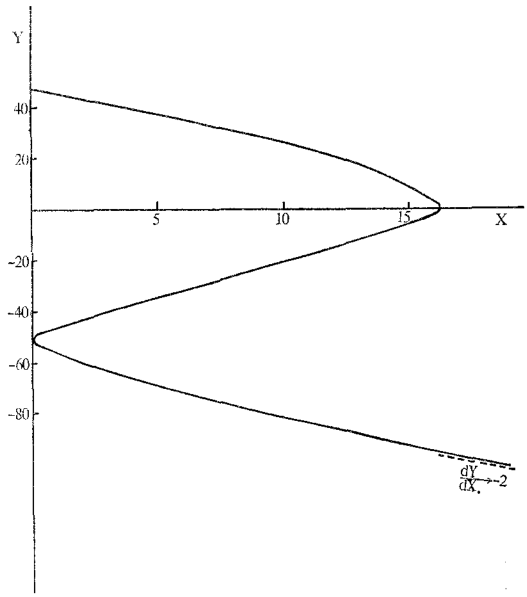

Figure 3.—The graph of $Y = \Delta'' - \frac{1}{2}\Delta'^2/\Delta$ against $X = \frac{1}{4}\Delta'^2/\Delta$ for the case $H_0 = K_0 = 0$, $\omega_0 = 5$. Since this case is symmetrical about the time origin the same $X - Y$ curve describes the evolution for $\tau > 0$ and $\tau < 0$.

exploration of equation (1.36). We plot only the region $\tau > 0$, since the present case is symmetrical yielding the same $X - Y$ curve for $\tau < 0$. The graph shows that Narlikar was in error in believing that there were no models passing between a minimum ($X = 0$, $Y > 0$) and a maximum ($X = 0$, $Y < 0$) of Δ. In particular, curves II of Narlikar's Fig. 1 are incorrect in demonstrating models apparently moving from a minimum direct to a singularity (at which $dY/dX \to -2$). The curves must first pass to a maximum at $X = 0$, $Y < 0$ before moving to the singularity, as shown in the present example. On the other hand Narlikar's models I are possible (and were confirmed in our own investigation); they arise in cases where there is never sufficient spin relative to shear and gravitation to create a minimum in Δ, nor sufficient velocities for escape to infinity. Hence there is one

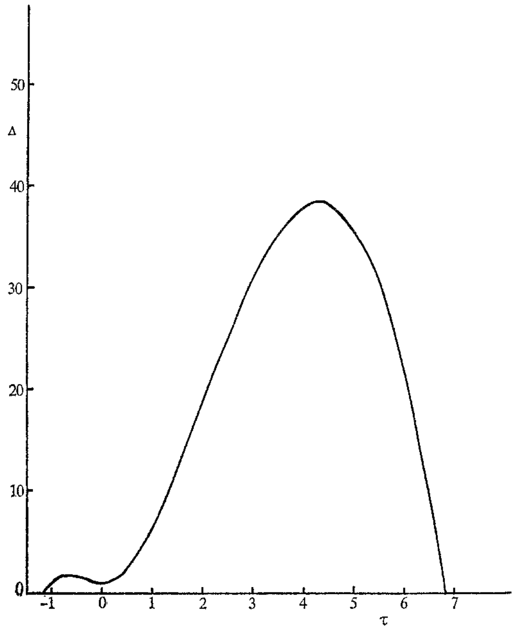

Figure 4.—The $\Delta - \tau$ relation for $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 2{\cdot}5$.

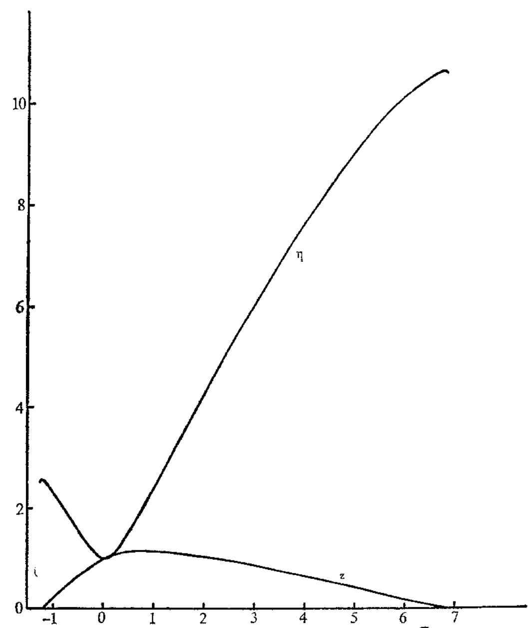

Figure 5.—The graphs of η and z against τ for the case $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 2{\cdot}5$.

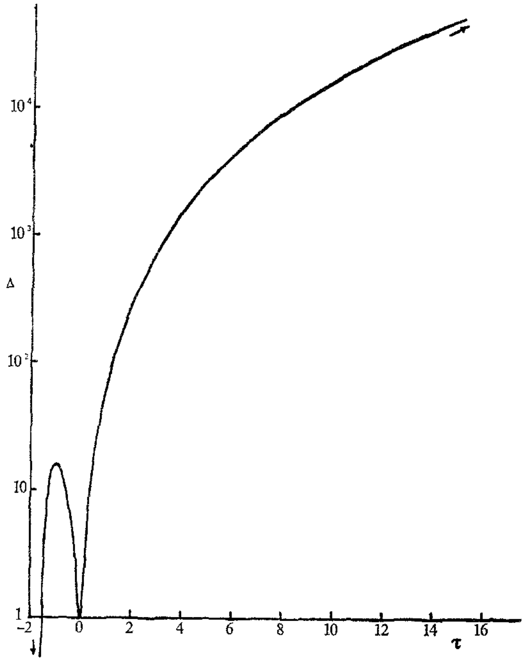

Figure 6.—The computation of Δ against τ for the case $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 7$ with Δ scaled logarithmically. An arrow indicates indefinite extension of the curve in that direction.

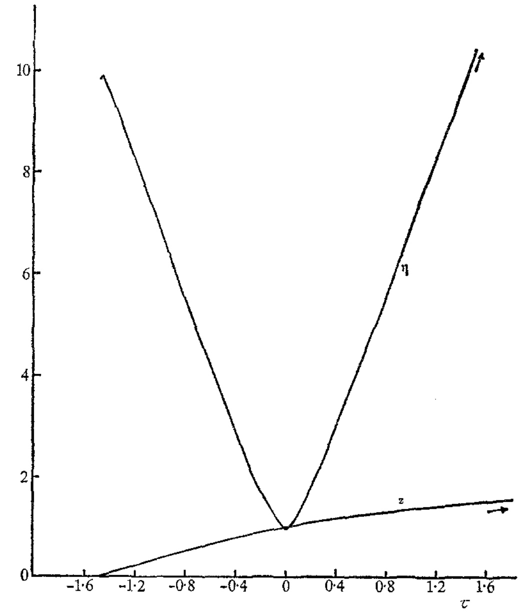

Figure 7.—The $\eta - \tau$ and $z - \tau$ relations for $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 7$.

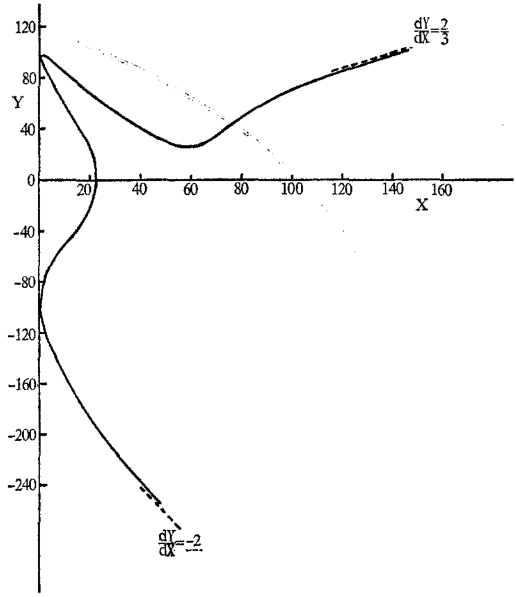

Figure 8.—The $X - Y$ relation for $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 7$.

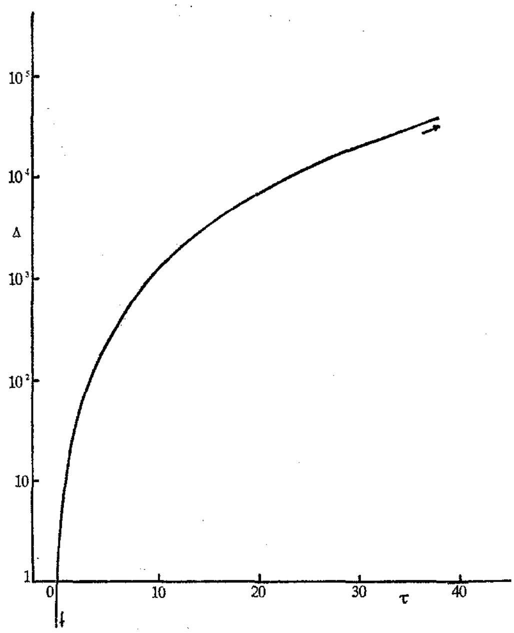

Figure 9.—The graph of Δ against τ for the case $H_0 = 2$, $K_0 = 1$, $\omega_0 = 5 \times 10^{-4}$.

maximum of Δ and two singularities in the complete development of such cases.

(b) $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 2{\cdot}50$

Two unequal maxima of Δ feature in this model which has non zero shear at the minimum lying at $\tau = \tau_0 = 0$ between the two maxima. In this case equation (2.39) of Section 2 holds so that there are two singularities. Although the material has a positive z motion at the minimum it is

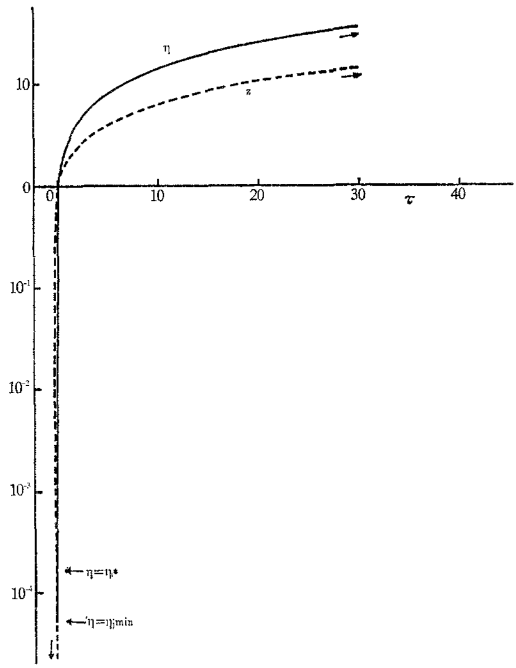

Figure 10.—Graphs of η against τ (full curve) and z against τ (broken curve) for the case $H_0 = 2$, $K_0 = 1$, $\omega_0 = 5 \times 10^{-4}$. The value $\eta = \eta_*$ on the η curve is the singularity value. The singularity value of z is zero.

insufficient for escape to infinity and final collapse follows. Once again as the singular state is approached the η expansion is checked by the gravitation field and flips over to a short contraction before $z \to 0$. The computed $\Delta - \tau$ development is shown in Figure 4, the $z - \tau$ and $\eta - \tau$ relations in Fig. 5.

(c) $H_0 = -0{\cdot}25$, $K_0 = 0{\cdot}50$, $\omega_0 = 7$

Here there is a maximum of Δ followed by a minimum where the spin relative to shear and gravitation is sufficiently great combined with the z motion to carry the material to indefinite expansion. The relevant expansion criterion is equation (2.41) in this case. The $\Delta - \tau$ computation appears in Fig. 6 where the scale of Δ is logarithmic. The $z - \tau$ and $\eta - \tau$ curves are shown in Fig. 7. In Fig. 8 we give the $X - Y$ relation for this model; after the Δ minimum ($X = 0$, $Y > 0$) the $X - Y$ curve maintains $Y > 0$ and is finally asymptotic to a gradient $dY/dX = 2/3$ where $\Delta \to \infty$ like τ^3. This illustrates another type of $X - Y$ curve not found by Narlikar in his numerical exploration of the $X - Y$ differential equation.

(d) $H_0 = 2$, $K_0 = 1$, $\omega_0 = 5 \times 10^{-4}$

Here $2H_0 + K_0 \neq 0$ so that the epoch $\tau = \tau_0$ is not a stationary value of Δ In fact the two turning points of Δ in this model occur extremely close to the singularity. Since $K_0 - H_0 < 0$ the possible singularities are at $\pi - \alpha$ and $-\alpha$, where in this case $\tan \alpha = \omega_0 = 5 \times 10^{-4}$ giving $\alpha \sim 5 \times 10^{-4}$ rad. Thus integration backwards in τ from $\tau = 0$ towards $\theta = \pi - \alpha$ leads to a singularity in analogy with Section 2(c)(i), setting $\pi - \alpha$ for α in all results there. Tracing forwards in τ-time from the singularity Δ first increases to a small maximum (equal to $5{\cdot}56 \times 10^{-10}$) and then drops down to a minimum (equal to $2{\cdot}63 \times 10^{-10}$). Both these turning points occur within a τ-time 10^{-4} from the singularity. The graph of Δ then climbs steadily to $\Delta = 1$ at $\tau = 0$. Integrating forwards from $\tau = 0$ this is a case when the analogue of (2.41) occurs, i.e. putting $\pi - \alpha$ for α, so that $z \to \infty$ when $\theta \to \theta_1 > -\alpha$. Turning to the radial coordinate η first rises slightly from its 'pancake' value $\eta_* = 1{\cdot}59 \times 10^{-4}$ at the singularity and then, again within a τ-time of 10^{-4}, passes through a maximum, quickly followed by a minimum (equal to $5{\cdot}13 \times 10^{-5}$). It then increases monotonically to infinity. The computed graphs of Δ, η, z against τ appear in Figs. 9, 10; their scale however is insufficient to show the above details of behaviour near the singularity.

(e) *Comparison with the Actual Universe*

In the preliminary account of this work (Davidson & Evans, 1971) model (c) was compared with the universe. This gave a satisfactory matching of present isotropy, Hubble time and the estimated age of the universe. The present spin was found to be $\sim 10^{-12}$ rad yr^{-1} which is certainly smaller than the upper limit 7×10^{-11} rad yr^{-1} deduced by Kristian & Sachs (1966) by direct observation of galaxies. However it is considerably higher than the upper limit of about 10^{-15} or 10^{-16} rad yr^{-1} (depending on the cosmological parameters) which has been construed by Hawking (1969) to follow from the limits of anisotropy of the 3°K microwave radiation.

The validity of Hawking's reasoning is a difficult matter to assess. If the radiation shared in the tendency to isotropy which we have found characteristic of the Newtonian ever expanding models, then his assumption that it retains the anisotropy of matter at last scattering would be invalid. But in any case we shall now make use of model (d) to show that a satisfactory matching of the present properties of the universe can be achieved, as well as a very much lower value for the present spin.

To identify the present epoch $t = t_1$ in the actual universe with an epoch $\tau = \tau_1$ in a model requires that the model should have come close to a state of isotropic expansion at $\tau = \tau_1$. Also the average of the radial and axial rates of strain in t time must be identified with the Hubble constant T^{-1}. That is, in terms of τ derivatives

$$\frac{1}{2}\left(\frac{z'}{z} + \frac{\eta'}{\eta}\right)_1 = \{(\tfrac{4}{3}\pi G\rho_0)^{1/2} T\}^{-1} \tag{3.1}$$

Hence

$$\frac{1}{2}\left(\frac{z'}{z} + \frac{\eta'}{\eta}\right)_1 \Delta_1^{1/2} = \{(\tfrac{4}{3}\pi G\rho_1)^{1/2} T\}^{-1} \tag{3.2}$$

where ρ_1 is the present mass density in the model. In this paper we take $T = 1{\cdot}3 \times 10^{10}$ yr and identify ρ_1 with the conservative value $1{\cdot}2 \times 10^{-30}$ g cm^{-3}. Hence for a satisfactory comparison with the universe at the present epoch on this basis a model must have at $\tau = \tau_1$

$$\frac{1}{2}\left(\frac{z'}{z} + \frac{\eta'}{\eta}\right)_1 \Delta_1^{1/2} = 4{\cdot}25 \tag{3.3}$$

and a sufficiently close equality of (z'/z), (η'/η) to achieve the observed degree of isotropy.

Analysis of model (d) shows that at $\tau = 30$, $z'/z = 0{\cdot}028$, $\eta'/\eta = 0{\cdot}033$, each of which is less than 10% from their average. This is well within the isotropy observed in measurements of Hubble's constant. Also at that epoch the model gives $\Delta_1 = 1{\cdot}88 \times 10^4$ so that the criterion (3.3) is very closely satisfied. For the density $\rho_0 = \rho_1 \Delta_1$ at the reference epoch $\tau = 0$ we therefore get the value $2{\cdot}26 \times 10^{-26}$ g cm^{-3}. This means that a unit of τ time has to be identified in t time with $1{\cdot}26 \times 10^{16}$ s or $3{\cdot}99 \times 10^8$ yr. Hence the age of the model universe between the singularity (which occurred at $\tau_* = -0{\cdot}407$) and the present epoch ($\tau = 30$) is

$$a = 1{\cdot}23 \times 10^{10} \text{ yr} \tag{3.4}$$

in satisfactory agreement with all astronomical evidence.

The value of η in the model at the epoch $\tau = 30$ was found to be $\eta_1 = 36{\cdot}30$. Hence the present spin of the universe would be

$$\omega_1 = \frac{\omega_0}{{\eta_1}^2} = \begin{cases} 3{\cdot}80 \times 10^{-7} \text{ rad/unit } \tau \text{ time, or} \\ 9{\cdot}35 \times 10^{-16} \text{ rad yr}^{-1} \end{cases} \tag{3.5}$$

This is of the order of the upper bound deduced by Hawking from the observations of the microwave radiation. We emphasise that models with even lower spin at the present epoch could easily be found, without removing the characteristic rotating 'pancake' origin of the Newtonian spinning models.

In this connection it should be noted that for the model (d) under discussion the spin at the singularity is very high compared with, say, a model of type (c) whose spin at the identified present epoch is so much higher. This is because relative to states of comparable density the radial shrinkage as the singularity is approached in the (d) model is in marked contrast to the (c) model where there is actually a prolonged radial expansion (apart from the small flipover at the last moment). Thus in the (d) model at the singularity $\eta_* = 1{\cdot}59 \times 10^{-4}$ so that at the moment of singularity the spin has the value

$$\omega_* = 4{\cdot}96 \times 10^{-5} \text{ rad yr}^{-1} \tag{3.6}$$

even although the total angle turned through by the universe in its $\sim 10^{10}$ yr history is less than π radians. As a comparison the present spin of the collapsed Galaxy is $\sim 10^{-7}$ rad yr^{-1}.

If a spin such as (3.6) was realised in our early universe it could not fail to affect fundamentally the nature of its history. We note the omnipresence of spin in objects in our observed universe ranging from planets, stars and pulsars to galaxies, with evidence of rotation in clusters and superclusters of galaxies. It would therefore seem naive to suppose that on an even larger scale spin was somehow unimportant despite the overwhelming evidence of a high density state in the past. This would seem especially relevant if it were the case that our observable universe had suffered a gravitational collapse in its prehistory. At the same time, if the collapsed state were of the nature of a rotating disc before the big-bang developed the theoretical problems would immediately seem more tractable than those of the point singularity in isotropic models.

References

Davidson, W. and Evans, A. B. (1971). *Nature, London*, **232**, 29.
Hawking, S. (1969). *Monthly Notices of the Royal Astronomical Society*, **142**, 129.
Heckmann, O. and Shucking, E. L. (1955). *Zeitschrift für Astrophysik*, **38**, 95; ibid. (1956) **40**, 81.
Kristian, J. and Sachs, R. K. (1966). *Astrophysical Journal*, **143**, 379.
McCrea, W. H. and Milne, E. A. (1934). *Quarterly Journal of Mathematics*, **5**, 73.
Narlikar, J. V. (1963). *Monthly Notices of the Royal Astronomical Society*, **126**, 203.
Raychaudhuri, A. (1955). *Physical Review*, **98**, 1123.
Shikin, I. S. (1971). *Soviet Physics: JETP*, **32**, 101.
Zeldovich, Y. B. (1965). *Soviet Astronomy-AJ*, **8**, 700.